EXTREMAL STRUCTURE OF WELL-CAPPED CONVEX SETS

BY

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Introduction. The Kreĭn-Milman Theorem states that a compact convex subset of a locally convex topological vector space (LCTVS) is the closed convex hull of its extreme points. Klee [5] has generalized this by proving that if X is a locally compact closed convex subset of a LCTVS, and X contains no line, then X is the closed convex hull of its extreme points and extremal rays.

Another approach to extending the Kreĭn-Milman Theorem, due to Choquet [1], uses the concept of a cap of a convex cone. A set C is said to be a *cap* of the cone P if C is a compact convex subset of P for which $P\backslash C$ is also convex. If each point of P is contained in a cap, then P is said to be *well-capped.* Choquet showed that if a closed convex cone in a LCTVS is well-capped, then it is the closed convex hull of its extremal rays.

These two approaches can be unified by extending the definition of cap to apply to arbitrary closed convex sets. Thus a *cap* C of a closed convex set X is a compact convex subset of X such that $X\backslash C$ is convex. The set X is *well-capped* if it is the union of its caps. Klee's techniques in [5] can be applied to show that a locally compact closed convex set containing no line is well-capped and that any closed convex well-capped subset of a LCTVS is the closed convex hull of its extreme points and extremal rays.

It follows from the above that the statement of a Kreĭn-Milman type theorem for well-capped sets includes the Klee and Choquet versions simultaneously. This is the principal motivation for the study of well-capped sets. It will be shown that each closed convex well-capped set generates (in a canonical way) a closed convex well-capped cone. In this sense the class of well-capped sets is closely related to the class of well-capped cones. A detailed discussion of well-capped cones can be found in [8].

In §1 some of the basic algebraic facts about caps of arbitrary closed convex sets are discussed. The notion of the cap boundary is introduced and is used to provide a convenient description of a cap by means of a certain affine functional.

In §2 the properties of well-capped sets and cones mentioned above are developed. §3 deals with various positive cones of measures on a locally compact Hausdorff space, and criteria for a measure to be contained in a cap are given. Meyer [7] has shown that if T is a locally compact, σ-compact Hausdorff space, with $C_\infty(T)$

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the space of continuous functions on T with compact support, then the cone of positive regular Borel measures on T is well-capped in the weak topology induced by $C_\infty(T)$. It is shown that if μ is a positive regular Borel measure on the locally compact Hausdorff space T such that either the support of μ is σ-compact or $\mu(T)$ is finite, then μ is contained in a cap of the cone of positive regular Borel measures. It is shown further that if T is a paracompact, locally compact Hausdorff space then μ is contained in a cap if and only if the support of μ is σ-compact.

§4 contains examples of closed convex sets which fail to be well-capped. Theorem 4.1 shows the existence of a closed convex set which is not well-capped, but is the vector sum of a universally capped cone (a cap C of P is called a *universal cap* of P if $P=\bigcup_{n=1}^{\infty} nC$) and a compact convex set. Thus the property of being well-capped is not preserved under set addition, even in this weak form. The remainder of the section is devoted to showing that several of the standard ordered topological vector spaces have positive cones which fail to be well-capped. They include the L^p spaces, $1<p\leqq\infty$, in their weak* topologies, and some of the frequently encountered function spaces.

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1. Let E be a Hausdorff locally convex topological vector space (LCTVS). The following notation will be used throughout: (x, y) denotes the open interval connecting the points x and y in E, $[x, y)$ the closed-open interval, etc. Let $)x, y($ denote the line in E determined by x and y. The open ray determined by x and y with end-point x is denoted by $(x, y($ and the corresponding closed ray is denoted by $[x, y($.

Let X be a closed convex subset of E.

Definition. A subset S of X is *extremal* if whenever $x, y \in X$ and (x, y) $S\neq\phi$, then $[x, y]\subset S$.

In working with well-capped sets, it is of considerable technical convenience to introduce a certain affine functional associated with a given cap. The purpose of this section is to construct this functional (see Theorem 1.6).

Definition. Let C be a cap of X. The *cap boundary* B_C of C in X is $\{x \in C \mid$ there exist x_1, x_2 such that $x \in (x_1, x_2)$; $x_1 \in C$, $(x, x_2]\subset X\backslash C\}$.

It is readily seen that B_C is convex and that B_C is empty if and only if C is extremal in X. Let S_C denote the affine variety spanned by C. If B_C is not empty let L_C be the affine variety spanned by B_C. Since C and B_C are convex,

$$S_C = \bigcup \{(x, y(\mid x, y \in C\} \quad \text{and} \quad L_C = \bigcup \{(x, y(\mid x, y \in B_C\}.$$

Proposition 1.1. (i) *If C is an extremal cap of X, then $S_C \cap X=C$.*

(ii) *If C is any cap of X, then $S_C \cap X$ is extremal in X.*

(iii) *If S is an extremal subset of X and C is a cap of S, then C is a cap of X. In particular if C is a cap of X and C' is a cap of $S_C \cap X$, then C' is a cap of X.*

Proof. (i) If $x \in (S_C \cap X)\backslash C$ then $x \in (y, z(; y, z \in C$. But then $z \in (y, x)$ so that $B_C \neq \phi$.

(ii) If C is extremal, the conclusion follows from (i). Suppose that $B_C \neq \phi$ and that $x \in (x_1, x_2) \cap S_C$; $x_1, x_2 \in X$. If $x \in C$ then either x_1 or x_2, say x_1, is in C. But then $x_2 \in (x_1, x(\subset S_C$, so $x_1, x_2 \in S_C$. If $x \in (S_C \cap X)\backslash C$ then $x \in (y, z(; y, z \in C$. Since $x \in (x_1, x_2)$ and $z \in (y, x)$, we have $z \in (z_1, z_2)$ where $z_i \in (y, x_i)$; $i=1, 2$. But then $z \in C$ implies z_1 and z_2 are in S_C. Thus $x_i \in (y, z_i(\subset S_C$.

(iii) Suppose x_1 and x_2 are in $X\backslash C$. If $(x_1, x_2) \cap C \neq \phi$ then $x_1, x_2 \in S$ since S is extremal. But $S\backslash C$ is convex so that either x_1 or x_2 is in C. Thus $[x_1, x_2] \subset X\backslash C$.

Assume now that C is a cap of X and that the cap boundary B_C is not empty.

Lemma 1.2. *Suppose z is contained in C.*

(i) *If $z \in L_C$, then $(z, y] \subset X\backslash C$ for all $y \in X\backslash C$.*

(ii) *If $z \in C\backslash L_C$, then $(z, y) \cap B_C \neq \phi$ for all $y \in (S_C \cap X)\backslash C$.*

Proof. (i) Assume first $z \in B_C$ and without loss of generality take $z=0$. Thus there is an $x_1 \in C$ such that $(0, -x_1] \subset X\backslash C$. Given $y \in X\backslash C$ we must show that $\lambda_0 y \in X\backslash C$; $0<\lambda_0<1$. There exists an η_0, $0 \leqq \eta_0 < 1$, such that $\eta y+(1-\eta)x_1 \in X\backslash C$ for all η, $1 \geqq \eta > \eta_0$. Then if $x=\eta y+(1-\eta)x_1$ with $1>\eta>\eta_0$ and η sufficiently close to 1, $x \in X\backslash C$ and the ray $(x, \lambda_0 y($ intersects $(0, -x_1)$ at some point w. Then $\lambda_0 y \in (x, w)$; $x, w \in X\backslash C$, so that $\lambda_0 y \in X\backslash C$.

If $z=0 \in L_C\backslash B_C$ then there exist x, η $(0<\eta<1)$ such that $x, \eta x \in B_C$. It is easy to verify that there exists $w \in (x, \lambda_0 y) \cap (\eta x, y)$. Since $\eta x \in B_C$, $w \in X\backslash C$. By the convexity of C, $\lambda_0 y \in X\backslash C$.

(ii) Again take $z=0$. Since $y \in S_C$, $y \in (x_1, x_0(; x_1, x_0 \in C$. Since $y \in X\backslash C$, x_0 can be chosen to be in B_C. It suffices to show there exists λ_0 $(0<\lambda_0<1)$ for which $\lambda_0 y \in C$. Suppose not, so that $(0, y] \subset X\backslash C$. There exists a $w \in (0, y)$ such that $x_0/2 \in (w, x_1)$. Furthermore, if $w' \in (x_0/2, w)$, then $w' \in (x_0, \lambda y)$ for some λ, $0<\lambda<1$. Thus by (i) $w' \in X\backslash C$. Thus $x_0/2 \in B_C$. But x_0 is in B_C also so that this implies $0 \in L_C$. Thus $\lambda_0 y \in C$ for some λ_0, $0<\lambda_0<1$.

Corollary 1.3. *If $z \in L_C \cap X$, then $z \in C$ and hence $L_C \cap X = L_C \cap C$.*

Proof. This follows from (i) and the fact that $L_C = \bigcup \{(x, y(\mid x, y \in B_C\}$.

Corollary 1.4. *The sets $C\backslash L_C$ and $(S_C \cap X)\backslash C$ are nonempty.*

Proof. This follows from (i) and the fact that B_C is assumed to be nonempty.

Corollary 1.5. *Assume $0 \in C\backslash L_C$. Then $S_C = \langle L_C \rangle$, the linear subspace generated by L_C, and hence L_C is a hyperplane in S_C disjoint from the origin.*

Proof. Clearly $\langle L_C \rangle \subset S_C$. To show the reverse inclusion it suffices to show $C \subset \langle L_C \rangle$. If $x \in C \cap L_C$, then $x \in \langle L_C \rangle$. If $x \in C\backslash L_C$, then by (ii) there exists x_0, y such that $x_0 \in B_C$, $y \in (S_C \cap X)\backslash C$ and $x_0 \in (x, y)$. Also by (ii) there is an $x_1 \in (0, y) \cap B_C$. Thus $x_0 = \lambda x + (1-\lambda)y$; $0<\lambda<1$, and $x_1 = \mu y$; $0<\mu<1$. Then

$$x = x_0/\lambda - [(1-\lambda)/\lambda]y = x_0/\lambda - [(1-\lambda)/\lambda](x_1/\mu) \in \langle L_C \rangle.$$

We now formulate the main result of this section.

THEOREM 1.6. *Let C be a cap of X with a nonempty cap boundary (equivalently, C is not extremal in X) and let S_C be the affine variety spanned by C. There exists an affine functional f on S_C such that*

(i) $C=\{x \in S_C \cap X \mid f(x) \leqq 1\}$,

(ii) $C_r=\{x \in S_C \cap X \mid f(x) \leqq r\}$ *(r any real number) is either empty or a cap of X,*

(iii) *f is lower-semicontinuous on $S_C \cap X$,*

(iv) *f is bounded below on $S_C \cap X$ and $C'=\{x \in S_C \cap X \mid f(x)=\inf f(S_C \cap X)\}$ is a nonempty extremal cap of X contained in $C \backslash L_C$.*

(v) *If g is any other affine functional on S_C satisfying* (i) *then $g=\alpha f+(1-\alpha)$, where $\alpha>0$. Thus the family of caps $\{C\}_{r \in R}$ and in particular the cap C' (as in* (iv)) *is independent of the choice of f.*

(vi) *If $0 \in C \backslash L_C$, then S_C is a subspace and there is a unique linear functional on S_C satisfying* (i); *if $0 \in C'$, then there exists a unique linear functional on S_C bounded below by* 0 *on $S_C \cap X$ and satisfying* (i).

Proof. Corollary 1.4 guarantees that $C \backslash L_C$ is not empty. It is easiest to consider first the case where $0 \in C \backslash L_C$. By Corollary 1.5, there is a unique linear functional f on S_C such that $f^{-1}(1)=L_C$. Choose $x_0 \in B_C$ and $\lambda_0>1$ such that $(x_0, \lambda_0 x_0] \subset X \backslash C$. To show f satisfies (i), let $x \in C$. If $x \in C \backslash L_C$, there exists (by 1.2) $z \in B_C$ such that $z=(1-\lambda)x+\lambda(\lambda_0 x_0)$, $0<\lambda<1$. Since $z, x_0 \in L_C$,

$$f(x)=f[(z-\lambda\lambda_0 x_0)/(1-\lambda)]=(1-\lambda\lambda_0)/(1-\lambda)<1.$$

Thus $C \subset \{x \in S_C \cap X \mid f(x) \leqq 1\}$. Now let $x \in S_C \cap X$ and suppose $f(x) \leqq 1$. To show $x \in C$ it suffices to show that $[x, \lambda_0 x_0) \cap L_C \neq \phi$. We have $x=r x_0+y$, where $r \leqq 1$ and $f(y)=0$. If $\alpha=(\lambda_0-1)/(\lambda_0-r)$, then $0<\alpha \leqq 1$ and $x_0+\alpha y=\alpha x+(1-\alpha)(\lambda_0 x_0)$. Since $f(x_0+\alpha y)=1$, $x_0+\alpha y \in L_C$. Thus $x_0+\alpha y \in L_C \cap X=L_C \cap C$.

To prove (ii) it suffices to show C_r is compact, for then the linearity of f and Proposition 1.1 serve to show C_r is a cap of X. Suppose first $r \geqq 1$. Then it is easily seen that $C_r=(rC) \cap X$. If $r<1$, then $C_r=[\lambda_0 x_0+\lambda(C-\lambda_0 x_0)] \cap X$, where $\lambda=(\lambda_0-r)/(\lambda_0-1)$. In either case C_r is compact.

Since $C_r=\{x \in S_C \cap X \mid f(x) \leqq r\}$ is compact it follows that f is l.s.c. on $S_C \cap X$. This proves (iii).

Since f is l.s.c. on the compact set C, f assumes its greatest lower bound at some point $x \in C$. Say $f(x)=a$. Since $0 \in C$, $a \leqq 0$. If $y \in S_C \cap X$ and $f(y) \leqq a$ then $y \in C$ and hence $f(y)=a$. Thus $a=\inf f(S_C \cap X)$. Now $C'=\{x \in S_C \cap X \mid f(x)=a\}$ is seen to be an extremal cap of X and since $a \leqq 0$, $C' \subset X \backslash L_C$. If $0 \in C'$, then f is bounded below on $S_C \cap X$ by 0.

To show (v) suppose g is an affine functional on S_C satisfying (i). If $x \in B_C$, then $g(x) \leqq 1$. But $x=(x_1+x_2)/2$; $x_1 \in C$ and $(x, x_2] \in X \backslash C$. So, if $g(x)<1$ then $(x, x_2] \cap \{x \in S_C \cap X \mid g(x) \leqq 1\} \neq \phi$. Thus $g(x)=1$ and hence $g(L_C)=\{1\}$. If $x \in C \backslash L_C$ it follows from Lemma 1.2 (ii) that $g(x)<1$. Thus $g^{-1}(1)=f^{-1}(1)=L_C$ so that $g=\alpha f+(1-\alpha)$. Since $\{x \in S_C \mid g(x)\leqq 1\}=\{x \in S_C \mid f(x)\leqq 1\}$, $\alpha>0$.

This completes the proof in case $0 \in C\backslash L_C$. The general case follows easily by considering an appropriate translation of X.

DEFINITION. If C is a cap of X with $B_C \neq \phi$ then any f satisfying (i) will be referred to as an *associated functional* of C.

2. The main results about well-capped cones are extended to well-capped sets in this section. In his discussion of locally compact closed convex sets, Klee [5] does not explicitly use the concept of a cap. However, his techniques can be readily adapted to prove that locally compact convex sets containing no lines are well-capped and that well-capped sets satisfy a Kreĭn-Milman type theorem. This is done in Theorems 2.1 and 2.2 below. It is also proved (Theorem 2.6) that the closed cone in $E\times R$ generated by $X\times\{1\}$ is well-capped if and only if X is a well-capped subset of E. Some related results are given by Hinrichen and Bauer in [4] in which they extend the results of Klee [5] to countable projective limits of locally compact convex sets. They also give an analcg to Theorem 2.6 where "well-capped" is replaced by "locally compact".

DEFINITION. A cap C is said to be a *universal cap* of X is the affine variety S_C spanned by C contains X.

If X has a universal cap C, then X is well-capped. For, if C is extremal, then $X=S_C \cap X=C$ (by 1.1(i)). Otherwise let f be an associated functional on S_C. Since $X\subset S_C$ each point of X is contained in some $C_r=\{x \in S_C \cap X \mid f(x)\leqq r\}$ and hence X is well-capped.

THEOREM 2.1. *If X is a locally compact closed convex subset of the LCTVS E such that X contains no line, then X has a universal cap.*

Proof. Assume $0 \in X$. By Proposition 3.2 of [5] there is a continuous linear function f on E such that $C_r=\{x \in X \mid f(x)\leqq r\}$ is compact for every real number r. Since f is linear and defined on all of E, C_r, with r sufficiently large, is clearly a universal cap of X.

THEOREM 2.2. *If X is a closed convex well-capped subset of a LCTVS E, then X is the closed convex hull of its extreme points and extremal rays.*

Let $\operatorname{ext}(X)$ denote the set of extreme points of X and $\operatorname{extr}(X)$ the union of extremal rays of X. The proof of 2.2 rests on the following simple lemma:

LEMMA 2.3. *Let X be a closed convex set and let f be an affine functional on X such that $x_0 \in \operatorname{ext}\{x \in X \mid f(x)=r\}$. Then x_0 lies on an extremal line segment or an extremal ray.*

Proof. (See also Theorem 3.4 of [5].) If $x_0 \notin \operatorname{ext}(X)$, then $x_0=(y+z)/2$, y, $z \in X\backslash\{x_0\}$. Since $x_0 \in \operatorname{ext}(f^{-1}(r) \cap X)$ it can be assumed that $f(y)<r$ and $f(z)>r$. If $]y, z[\cap X$ is not extremal there exist $y', z' \in X\backslash]y, z[$ such that $x_0 \in (y', z')$.

Assume $f(y')<r$ and $f(z')>r$. Then there exist $x_1, x_2 \in f^{-1}(r) \cap X$ such that $x_1 \in (y, z')$, $x_2 \in (y', z)$ and $x_0 \in (x_1, x_2)$. But then $x_1 = x_0 = x_2$ and hence $y', z' \in]\!)y, z(\![\cap X$.

Proof of 2.2. Observe first that if Y is a closed convex set which contains a cap C, then ext $(Y) \neq \phi$. For, by 1.6 (iv) either C is extremal or C contains an extremal cap of Y. Thus by the Kreĭn-Milman theorem, ext $(Y) \neq \phi$.

Let $X' = \text{cl-conv}(\text{ext}(X) \cup \text{extr}(X))$ and suppose $x \in X \backslash X'$. An application of the separation theorem yields a continuous linear functional f on E such that $f(x) = \alpha < \inf f(X')$. If $Y = f^{-1}(\alpha) \cap X$, then Y is a closed convex subset of X and hence is well-capped. Let $x_0 \in \text{ext}(Y)$. By Lemma 2.3 there exist $y, z \in X$ such that $x_0 \in (y, z)$ and $]\!)y, z(\![\cap X$ is an extremal subset of X. Since X is well-capped it contains no lines and thus $]\!)y, z(\![\cap X$ is either a point, a closed line segment or a ray. In any case $x_0 \in \text{cl-conv}(\text{ext } X \cup \text{extr } X) = X'$. But $f(x_0) < \inf f(X')$. Thus it must be the case that $X = X'$.

Definition. If X is a convex subset of E let $P(X)$ (or P, if no confusion results) denote the *closed* cone in $E \times R$ generated by $X \times \{1\}$. It is easily seen that $P(X)$ is convex.

Proposition 2.4. *If $0 < r < s < \infty$ and U is an open set in E, then $(r, s) \cdot [U \times \{1\}] = \{(tu, t) \mid r < t < s, u \in U\}$ is open in $E \times R$.*

Proof. Let $R^+ = \{r \in R \mid r > 0\}$. The map $F: E \times R^+ \to E \times R^+$, defined by $F(x, r) = (rx, r)$ is a homeomorphism of $E \times R^+$ onto itself, taking the open set $U \times (r, s)$ onto $(r, s) \cdot [U \times \{1\}]$.

Definition (Choquet [1]). Let $x_0 \in X$, where X is a closed convex set. The *asymptotic cone* $A(X)$ of X is $\bigcup \{[0, y(\mid x_0 + [0, y(\subset X\}$. It is easily seen that $A(X)$ is a closed convex cone such that $X + A(X) \subset X$. In particular, $A(X)$ is independent of the choice of $x_0 \in X$.

Proposition 2.5. *If X is a closed convex set then $(x, r) \in P(X)$ if and only if either*

(i) *$r > 0$ and $x/r \in X$, or*

(ii) *$r = 0$ and $x \in A(X)$.*

Proof. If $r > 0$ and $x/r \in X$, then $(x, r) = r(x/r, 1) \subset r \cdot (X \times \{1\}) \subset P$. If $r = 0$ and $x \in A$, then $(\lambda x_0 + (1-\lambda)x, \lambda) = \lambda([x_0 + (1-\lambda)/\lambda]x, 1) \subset P$, where $x_0 \in X$. Since P is closed, letting $\lambda \to 0$, it is seen that $(x, 0) \in P$.

Now suppose neither condition holds for (x, r) and let $r \geqq 0$. Let P' denote the (not necessarily closed) cone generated by $X \times \{1\}$. If $r > 0$ and $x/r \notin X$ there is a neighborhood U of x/r in E such that $U \cap X = \phi$. Thus $(s, t) \cdot [U \times \{1\}]$, $0 < s < 1 < t$, is an open neighborhood of $(x/r, 1)$ in $E \times R$ disjoint from P'. Thus $(x, r) \notin P$.

Finally, we show that if $(x, 0) \in P$ then $x \in A$. If $x_0 \in X$ then for any $r \geqq 0$, $(x_0 + rx, 1) = (x_0, 1) + r(x, 0) \in P$. By (i) this implies $x_0 + rx \in X$ and hence $x \in A$.

Observe that the map $\Phi: E\times R \to E\times R$ defined by $\Phi(x, t)=(x+tx_0, t)$ is an isomorphism of $E\times R$ onto itself taking $P(X)$ onto $P(X+x_0)$, where X is any closed convex subset of E and $x_0 \in E$. Consequently translating X in E does not change any essential properties of $P(X)$ (in particular C is a cap of $P(X)$ if and only if $\Phi(C)$ is a cap of $P(X+x_0)$). Thus the location of the origin in E can be chosen for the greatest convenience.

THEOREM 2.6. *If X is a closed convex subset of a LCTVS E then $P(X)$ is well-capped if and only if X is well-capped. Also, $P(X)$ has a universal cap if and only if X has a universal cap.*

Proof. If P is well-capped, then X, a closed convex subset of P, is also well-capped. Suppose X is well-capped and let $(x_0, r_0) \in P$. If $r_0=0$ then $(x_0, 0) \in A(X) \times\{0\}$. Since $A(X)\times\{0\}$ is a well-capped extremal subset of P, Proposition 1.1 shows $(x_0, 0)$ is contained in a cap of P. If $r_0>0$ it suffices to find a cap C of P containing $(x_0/r_0, 1)$. Thus, assume to begin with that $r_0=1$. In this case $x_0 \in X$ so that there is a cap C_1 of X containing x_0. Using the observation preceding the theorem it can be assumed the origin is situated as follows: If C_1 is extremal, let $0 \in C_1$. Otherwise let f be an associated functional on the affine variety $S(C_1)$ spanned by C_1 and let $0 \in \{x \in C_1 \mid f(x)=\inf f(C_1)\}$. Thus f can be chosen to be linear and bounded below by 0 on $S(C_1) \cap X$. In either case $S(C_1)$ is a linear subspace. By 1.1, $S(C_1) \cap X$ is an extremal subset of X. Let $S=(S(C_1)\times R) \cap P(X)$. It is not difficult to show S is an extremal subset of $P(X)$ containing $C_1\times\{1\}$.

Suppose now C_1 is an extremal cap of X. Then $S=P(C_1)=[0, \infty)[C_1\times\{1\}]$ and $C=[0, 1][C_1\times\{1\}]$ is a cap of $P(C_1)$ containing $C_1\times\{1\}$. Since $P(C_1)$ is extremal in P, C is a cap of P. If C_1 is a universal extremal cap, then $C_1=X$ and C is a universal cap of P. If C_1 is not extremal in X, then define F on $S(C_1)\times R$ by $F(x, r) =(f(x)+r)/2$, where f is the associated functional of C_1, and let

$$C = \{(x, r) \in S \mid F(x, r) \leqq 1\}.$$

Clearly $C_1\times\{1\}\subset C$. Since F is linear on $S(C_1)\times R$ and $S=[S(C_1)\times R] \cap P$ is extremal it follows from 1.1 that C is a cap, providing C is compact. Since f is l.s.c. on $S(C_1) \cap X$, f is l.s.c. on $S(C_1) \cap 2X$. Thus f is l.s.c. on $2C_1$ so that F is l.s.c. on the compact set $K=(2C_1\times[0, 2]) \cap P\subset S$. Let $C'=\{(x, r) \in K \mid F(x, r)\leqq 1\}$. We will show $C=C'$ thus proving C is compact. Clearly $C'\subset C$. Suppose $(x, r) \in C$. Since $F(x, r)=(f(x)+r)/2\leqq 1$ and $f(x)\geqq 0$, $0\leqq r\leqq 2$. If $r=0$ then $x \in A$. Since $f(x)\leqq 2$, $x \in C_2 \cap A\subset 2C_1$, where $C_2=\{x \in S(C_1) \cap X \mid f(x)\leqq 2\}$. Suppose $r>0$. Now $f(x/r)\leqq(2-r)/r$ so that $x/r \in C_{(2-r)/r}$. If $0<r\leqq 1$ then $x \in rC_{(2-r)/r}\subset C_{2-r}\subset C_2 \subset 2C_1$. If $1\leqq r\leqq 2$ then $x \in rC_{(2-r)/r}\subset rC_1\subset 2C_1$. Thus $(x, r) \in C'$ and so $C=C'$.

If C_1 is a universal cap of X then $S=P(X)$ and it follows from the definition of C that C is a universal cap of $P(X)$. Also, if C is any universal cap of $P(X)$, then $rC \cap [X\times\{1\}]$ is a universal cap of X, for r sufficiently large.

Choquet has shown (see also [7] and [8]) that if $E=\prod_{n=1}^{\infty} E_n$, where E_n is a LCTVS and P_n is a well-capped cone in E_n, then $P=\prod_{n=1}^{\infty} P_n$ is a well-capped cone in E. The analogous result for well-capped sets can be stated as a corollary to this and to Theorem 2.6.

COROLLARY 2.7. *If $X_n \subset E_n$ is a closed convex well-capped set for each positive integer n, then $X=\prod_{n=1}^{\infty} X_n \subset E=\prod_{n=1}^{\infty} E_n$ is a well-capped subset of E.*

3. The most frequently encountered examples of well-capped cones are cones of positive measures on locally compact Hausdorff spaces. In this section the caps of various positive cones of measures are discussed.

Let T denote a locally compact Hausdorff space and let $C_0(T)$ be the space of all continuous real-valued functions on T which "vanish at infinity". That is, $C_0(T)$ consists of those f with the property that for each $\varepsilon>0$ there exists a compact subset B of T such that $x \notin B$ implies $|f(x)|<\varepsilon$. Then $C_0(T)$ is a Banach space in the sup norm and its dual $C_0(T)^*$ can be identified with the space of all finite regular Borel measures on T.

THEOREM 3.1. *If T is a locally compact Hausdorff space, then the positive cone P of $C_0(T)^*$ has a universal cap in the weak* topology.*

Proof. Let U denote the unit ball in $C_0(T)^*$. Let $C=\{\mu \in P \mid \mu(T)=\|\mu\| \leqq 1\}$ $=U \cap P$. The set C is a universal cap of P.

Let $C(T)$ be the space of all continuous real-valued functions on T with the topology of uniform convergence on compacta. The dual $C(T)^*$ can be identified with the space of all regular Borel measures with compact support [3]. The support of a positive measure μ is taken to mean $\bigcap \{F \subset T \mid F \text{ closed and } \mu(T \backslash F)=0\}$. The support of a signed measure is the support of its total variation. The measures in $C(T)^*$ are finite since they have compact support.

THEOREM 3.2. *If T is a locally compact Hausdorff space, then the positive cone P of $C(T)^*$ is well-capped in the weak* topology.*

Proof. Let B be a compact subset of T and let r be a positive real number. Then $P'=\{\mu \in P \mid \operatorname{supp} \mu \subset B\}$ is an extremal subset of P. Let $C=\{\mu \in P' \mid \mu(T) \leqq r\}$. Let $U=\{f \in C(T) \mid \sup_{x \in B} |f(x)| \leqq 1/r\}$. Then U is a neighborhood of 0 in $C(T)$ and C is a closed subset of $U^0=\{\mu \mid |\mu(f)| \leqq 1 \text{ for all } f \in U\}$. Thus C is compact and therefore a cap of P. Since every measure in P is contained in such a cap, P is well-capped.

Consider now $C_\infty(T)=\{f \in C(T) \mid \text{there exists } B \subset T, B \text{ compact and } \operatorname{supp} f \subset B\}$. One form of the Riesz Representation Theorem states that to each positive linear functional ϕ on $C_\infty(T)$ there corresponds a unique regular Borel measure μ on T such that $\phi(f)=\int f \, d\mu$ for all $f \in C_\infty(T)$. Let P be the cone of positive regular Borel measures on T.

It is convenient to represent P as a projective limit of well-capped cones of measures. Toward this end let $\mathscr{B}$ be a family of open subsets of T with compact closure (directed by inclusion) such that if C is any compact subset of T then $C \subset B$ for some $B \in \mathscr{B}$. Let $E_B = C_0(B)$ with the norm topology. Then E_B is isometrically isomorphic to $\{f \in C_\infty(T) \mid \operatorname{supp} f \subset \bar{B}\}$. Let E_B^* be the dual of E_B with the weak* topology. Then 3.1 shows the positive cone of E_B^* has a universal cap.

Let $E^* = \lim \operatorname{proj} \{E_B^* \mid B \in \mathscr{B}\} = \{F \in \prod_B E_B^* \mid F(B_1)(f) = F(B_2)(f)$ whenever $B_1 \subset B_2$ and $f \in E_{B_1}\}$. Then P with the weak topology induced by $C_\infty(T)$ is isomorphic to the positive cone of E^* with the relative product topology under the correspondence $\mu \to F_\mu$, where $F_\mu(B)(f) = \mu(f) = \int f \, d\mu$. The correspondence is 1-1 and onto by the Riesz Representation Theorem cited above. It follows easily from the definition of E^* that the correspondence is a homeomorphism.

Suppose now that T is locally compact and σ-compact. Meyer has shown that in this case the cone of positive regular Borel measures is well-capped by constructing a cap containing a given positive measure [7]. An alternative proof, implicit in [1], uses Choquet's theorem that a countable product of well-capped cones is well-capped.

THEOREM 3.3 (MEYER). *Let T be a locally compact, σ-compact Hausdorff space. Then the cone of positive regular Borel measures on T is well-capped.*

Proof. Since T is σ-compact, the collection $\mathscr{B}$ can be chosen to be countable. The positive cone P of the projective limit E^* is easily seen to be a closed subset of the positive cone of $\prod E_B^*$. Since this product is countable P is well-capped by 2.7. This shows that the cone of positive regular Borel measures is well-capped.

If T is not σ-compact then the cone of positive measures need not be well-capped. As an example let T be an uncountable discrete space. It will follow from Theorem 3.5 below that if μ is any measure on T whose support is not σ-compact then μ is not contained in a cap.

However, we can state the following theorem.

THEOREM 3.4. *If μ_0 is a positive regular Borel measure on the locally compact Hausdorff space T, and if either*

(a) $\mu_0(T) < \infty$, *or*

(b) $\operatorname{supp} \mu_0$ *is σ-compact,*

then μ_0 is contained in a cap of the cone P of positive regular Borel measures on T.

Proof. Let $P' = \{\mu \in P \mid \mu(T) < \infty\}$. Then P' is an extremal subset of P. Let $C = \{\mu \in P' \mid \mu(T) \leqq 1\}$. If $U = \{f \in C_\infty(T) \mid \sup |f(T)| \leqq 1\}$ then U^0 is a compact subset of $P' - P'$ in the weak topology induced by $C_\infty(T)$. But C is a closed subset of U^0 so that C is a universal cap of P'. Thus (a) follows.

If $\operatorname{supp} \mu_0$ is σ-compact let T' be the support of μ_0 and let $P' = \{\mu \in P \mid \operatorname{supp} \mu \subset T'\}$. Again P' is an extremal subset of P. Also P' is isomorphic to the cone of positive regular Borel measures on T' with the weak topology induced by $C_\infty(T')$. Thus by 3.4 P' is well-capped and therefore μ_0 is contained in a cap of P.

There may be measures on T which satisfy (a) but not (b). As an example consider the space T consisting of an uncountable product of unit intervals with one point of the product removed. Since an uncountable product of intervals is not first countable it follows easily that T is not σ-compact. If μ denotes the product of Lebesque measures, restricted to T, then μ is finite and the support of μ is T.

The following theorem shows that this cannot happen if T is paracompact.

THEOREM 3.5. *Let T be a paracompact, locally compact Hausdorff space. A measure μ in the cone P of positive regular Borel measures on T is contained in a cap of P if and only if the support of μ is σ-compact.*

Proof. We use the fact that a locally compact paracompact space is the union of disjoint open σ-compact subsets (see, e.g. Theorem 7.3 of [2]). Thus if X is a closed subset of T and X is not σ-compact then there exists an uncountable family $\mathscr{W}$ of pairwise disjoint open sets such that $W \in \mathscr{W}$ implies $W \cap X \neq \phi$.

Assume $\mu_0 \in P$ and supp $\mu_0 = X$ is not σ-compact. Suppose that $\mu_0 \in P$ is contained in a cap C of P. Let F be the linear functional associated with C (as in 1.6). Let $\mathscr{W}$ be a family of open sets as above. If $W \in \mathscr{W}$ then $W \cap X \neq \phi$ and therefore $\mu_0(W) > 0$. Let $\mu_W = \mu_0|_W$.

Note that $0 \leqq \mu_W \leqq \mu_0$ implies that $\mu_W \in C$. Since $\mu_W > 0$ and C is compact, $F(\mu_W) > 0$. If $W_1, \ldots, W_n \in \mathscr{W}$, then $\sum_{k=1}^n \mu_{W_k} \leqq \mu_0$. But $F(\sum_{k=1}^n \mu_{W_k}) = \sum_{k=1}^n F(\mu_{W_k}) \leqq 1$. On the other hand, since $\mathscr{W}$ is uncountable and $F(\mu_W) > 0$ for all $W \in \mathscr{W}$, the supremum of the above sums taken over all finite subsets of $\mathscr{W}$ is $+\infty$. This contradiction shows that no cap of P contains μ_0.

4. In this section some examples of sets and cones which fail to be well-capped are discussed. It would seem plausible that the vector sum $P+B$ of a well-capped cone P and a compact convex set B is well-capped, but the following example shows that this is not the case.

THEOREM 4.1. *There exist a closed convex cone P with a universal cap and a compact convex set B such that the closed convex set $X = P + B$ is not well-capped.*

Proof. Let T be an uncountable discrete space and let $E = l^1(T)$, with the weak* topology as the dual of $c_0(T)$. Let P be the positive cone of E and let B be the unit ball of E. By 3.1 P has a universal cap and B is a compact convex subset of E in the weak* topology.

Every element $x \in E$ can be written in the form $x = x^+ - x^-$, where $x^+(t) = x(t) \vee 0$ and $x^-(t) = -(x(t) \wedge 0)$. If $X = P + B$ then $X = \{x \in E \mid \sum_{t \in T} x^-(t) \leqq 1\}$.

We will show now that if C is any cap of X then C contains no nonzero element of P. For suppose $x_0 \neq 0$, and $x_0 \in C \cap P$. Since $C \cap P$ is a cap of P, $0 \in C \cap P \subset C$. Let S_C be the linear subspace generated by C and let $\delta_t(s) = \delta_{ts}$, the Kronecker delta.

For any $t \in T$ we can write $0 = r/(1+r)[-\delta_t] + 1/(1+r)[r\delta_t]$ where r is a positive number sufficiently large so that $r\delta_t \in X \backslash C$. This shows that $-\delta_t \in C$ and that

$\delta_t \in S_C$. Also the cap boundary B_C is not empty. Since $0 \in C \backslash L_C$ (L_C is the linear variety generated by B_C) by 1.6 there is a linear functional f on S_C such that

$$\{x \in S_C \cap X \mid f(x) \leqq 1\} = C.$$

Since C contains no rays f must be strictly positive on (the nonzero elements of) $P \cap S_C$. Let $a_t = f(\delta_t)$. Then $a_t > 0$ for all $t \in T$. We show next that $f(x) = \sum_{t \in T} a_t x(t)$, for all $x \in S_C \cap P$. If $x = \sum_{t \in T'} x(t)\delta_t$, where T' is a finite subset of T, then $x \in S_C$ and $f(x) = \sum_{t \in T} a_t x(t)$. Given any $x \in S_C \cap P$ let $y_{T'} = \sum_{t \in T'} x(t)\delta_t$, where T' is finite. Then with T' ranging over the finite subsets of T, $\{y_{T'}\}$, is a net in $S_C \cap P$ converging to x. Since f is l.s.c. on $S_C \cap P$, $f(x) \leqq \liminf f(y_{T'}) \leqq \sum_{t \in T} a_t x(t)$. Since f is positive on $P \cap S_C$ and $y_{T'} \leqq x$, $f(x) \geqq f(y_{T'})$, for all T'. Thus $f(x) \geqq \limsup f(y_{T'})$. This shows that $f(x) = \lim f(y_{T'}) = \sum_{t \in T} a_t x(t)$.

Since T is uncountable and each $a_t > 0$, there exists a positive integer N such that $T' = \{t \in T \mid a_t \geqq 1/N\}$ is uncountable. But now C cannot be closed. For, take $x = [(1+\alpha)/a_{t_0}]\delta_{t_0}$ where $0 < \alpha < 1/N$ and $t_0 \in T$. Then $x \in S_C$ and $f(x) = 1 + \alpha > 1$ so that $x \notin C$. Let $U = \{y \in E \mid |s_i(y) - s_i(x)| < \varepsilon, i = 1, \ldots, n\}$, where $s_1, \ldots, s_n \in c_0(T)$. U is a weak* neighborhood of x. Now $T'' = \{t \in T \mid s_i(t) \neq 0, \text{ some } i \in 1, \ldots, n\}$ is countable. Let $t' \in T' \backslash T''$. Let $y = x - \delta_{t'}$. Since $s_i(\delta_{t'}) = 0$, $i = 1, \ldots, n$, we have $s_i(y) = s_i(x)$ so that $y \in U \cap S_C$. But $f(y) = 1 + \alpha - a_{t'} \leqq 1$ so that $y \in C$. But U is an arbitrary neighborhood of x and $U \cap C \neq \phi$. Thus $x \in \bar{C} \backslash C$. This shows that no cap of X can contain x_0.

The positive cones of various L^p spaces are now investigated. Consider first the l^p spaces, $1 \leqq p \leqq \infty$.

Theorem 4.2. *The positive cone of l^p $(1 \leqq p \leqq \infty)$ is well-capped in the weak* topology if and only if $p = 1$.*

Proof. By 3.1 the positive cone of l^1 has a universal cap. Suppose now that $1 < p \leqq \infty$. Let $a = \{a_n\}_{n=1}^{\infty}$ be a sequence of positive numbers such that $a \in l^p \backslash l^1$. Suppose that $a \in C$, where C is a cap of the positive cone of l^p. Let f be the associated linear functional and assume without loss of generality that $f(a) = 1$. Let δ^n denote the sequence which is 1 in the nth place and 0 elsewhere. Since $a_n\delta^n \leqq a$, $a_n\delta^n \in C$. Thus $f(a_n\delta^n) = a_n f(\delta^n) > 0$. Let $b_n = f(\delta^n)$. Then $b_n > 0$. Let $s_n = \sum_{k=1}^{n} a_k \delta^k$. Since $s_n \to a$ in the norm topology it converges to a in the weak* topology. Since f is l.s.c. on C, $f(a) \leqq \liminf f(s_n)$. Also $s_n \leqq a$, for all n, so that $f(a - s_n) \geqq 0$. Thus $f(a) \geqq \limsup f(s_n)$ and consequently $f(a) = \lim f(s_n) = \sum_{n=1}^{\infty} a_n b_n < \infty$. This implies that there exists a subsequence $\{b_{n_k}\}$ of $\{b_n\}$ such that $\lim_{k \to \infty} b_{n_k} = 0$. For if not, there exist $\varepsilon > 0$ such that $b_n \geqq \varepsilon$, for all n. In this case $\sum a_n b_n \geqq \varepsilon \sum a_n = +\infty$, since $a \notin l^1$. So assume $\{b_{n_k}\}$ has been chosen so that $0 < b_{n_k} \leqq 1/k$, for all k. Then $f(k\delta^{n_k}) = kb_{n_k} \leqq 1$. This implies $k\delta^{n_k} \in C$. Since $\|k\delta_{n_k}\| = k$, C is unbounded in norm. But l^p is the dual of a Banach space so that if C is unbounded in norm it is also unbounded in the weak* topology. Thus C is not compact.

We now consider $L^p(X, \mathscr{S}, \mu)$ spaces where there exists a nonatomic measurable subset of X.

THEOREM 4.3. *Let $(X, \mathscr{S}, \mu)$ be a complete σ-finite measure space and suppose there exists an $S \in \mathscr{S}$ such that $0 < \mu(S) < \infty$ and S contains no atoms with respect to μ. Then the positive cone of $L^p(X, \mathscr{S}, \mu)$, $1 < p \leqq \infty$, in the weak* topology as the dual of $L^q(X, \mathscr{S}, \mu)$ $(1/p + 1/q = 1)$ is not well-capped.*

Proof. We show that no cap of the positive cone P of L^p contains $\mathscr{X}_S$, (the characteristic function of S). For suppose $\mathscr{X}_S \in C$, where C is a cap of P. Let F be the associated linear functional and assume $F(\mathscr{X}_S) = 1$. Then F defines a finite measure μ_F on the measurable subsets of S by $\mu_F(A) = F(\mathscr{X}_A)$. Since F is positive we have $\mu_F(A) \leqq \mu_F(B)$ whenever $A \subset B \subset S$. To see that μ_F is countably additive let $A = \bigcup_{n=1}^{\infty} A_n$, $\{A_n\}_{n=1}^{\infty}$ a pairwise disjoint collection of measurable subsets of S. Let $B_n = \bigcup_{k=1}^{n} A_k$. Then $\mathscr{X}_{B_n} \leqq \mathscr{X}_A \leqq \mathscr{X}_S \in C$ and $\mathscr{X}_{B_n} \to \mathscr{X}_A$ in the weak* topology. Since F is a positive functional l.s.c. on C, we have $\mu_F(A) = \lim F(\mathscr{X}_{B_n}) = \sum_{n=1}^{\infty} \mu_F(A_n)$.

Since S has no atoms, given a positive integer n we can choose n disjoint subsets $S_1, \ldots, S_n$ of S such that $\mu(S_k) = \mu(S)/n$, and $S = \bigcup_{k=1}^{n} S_k$. (This is by Liapunov's Theorem, see e.g. [6].) Now $1 = F(\mathscr{X}_S) = \sum_{k=1}^{n} F(\mathscr{X}_{S_k})$ so that there exists S_k such that $0 < F(\mathscr{X}_{S_k}) \leqq I/n$. Then $n\mathscr{X}_{S_k} \in C$ since $F(n\mathscr{X}_{S_k}) \leqq 1$. But $\|n\mathscr{X}_{S_k}\|_p = (\int_{S_k} n^p \, d\mu)^{1/p} = (n^p \mu(S)/n)^{1/p} = [\mu(S)]^{1/p} = n^{(p-1)/p}$, for $1 < p < \infty$, and $\|n\mathscr{X}_{S_k}\|_\infty = n$. Thus C is unbounded in norm and since L^p is the dual of a Banach space C is unbounded in the weak* topology and hence not compact.

THEOREM 4.4. *Let T be a completely regular Hausdorff space and let $E = C(T)$, the space of all continuous real-valued functions on T. Then the positive cone P of E is well-capped in the topology of pointwise convergence if and only if T is countable and discrete.*

Proof. Suppose P is well-capped. Given $x \in T$ choose an open neighborhood U_x of x and choose $f \in P$ such that $f(T) \subset [0, 1]$, $f(x) = 1$ and $f(T \backslash U_x) = \{0\}$. Let C be a cap of P containing f. For each open neighborhood U of x, $U \subset U_x$, there exists $g \in E$ such that $g(T) \subset [0, 1]$, $g(x) = 1$ and $g(T \backslash U) = \{0\}$. Let $f_U = f \wedge g$. Then $0 < f_U \leqq f$ so that $f_U \in C$. With the open neighborhoods of x directed down by inclusion $\{f_U\}$ is a net in C converging pointwise to δ_x, the characteristic function of $\{x\}$. Since C is compact, $\delta_x \in C \subset E$. But δ_x continuous implies $\{x\}$ is open in T. Since this applies to any x, T is discrete. Thus T is locally compact and P can be identified with the cone of positive regular Borel measures on T. But the fact that T is discrete means, in particular, that T is paracompact and by 3.5 P is well-capped if and only if T is σ-compact. But T is σ-compact if and only if T is countable.

If T is assumed to be locally compact and Hausdorff, then the same argument can be applied to $C_\infty(T)$.

THEOREM 4.5. *If T is a locally compact Hausdorff space, then the positive cone P of $C_\infty(T)$ is well-capped in the topology of pointwise convergence if and only if T is discrete.*

Proof. As in 4.4, if P is well-capped then T is discrete. But in this case, P can be identified with the cone of positive regular Borel measures with compact support. Then P is well-capped by 3.2.

Observe that if the positive cone of $C(T)$ or $C_\infty(T)$ fails to be well-capped in the topology of pointwise convergence, then it is certainly not well-capped in any stronger topology.

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